Hamiltonian reduction of cosmological perturbations

Jinn-Ouk Gong

Instituut-Lorentz for Theoretical Physics, Universteit Leiden
2333 CA Leiden, The Netherlands

KIAS, Seoul, Korea
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Based on

- JG and S. Koh, to appear soon
- JG and S. Koh, in preparation
- JG and M. Sasaki, in preparation

Outline

1. Introduction
   - Cosmological perturbations and gauge issue
   - Systematics of extracting physical modes
2. Formulation
   - Hamiltonian reduction
   - Gauge transformations
3. Examples
   - Canonical single field
   - Hořava-Lifshitz gravity
   - Beyond linear perturbation
4. Conclusions

Primordial cosmological perturbations

What we understand regarding primordial cosmological perturbations:

- Nearly scale invariant spectrum
- Nearly Gaussian distribution
- Origin of large scale structure
- (Probably) produced during inflation

In agreement with recent observations: WMAP, SDSS, etc

Gauge degrees of freedom

General relativity: choice of coordinate system → gauge theory

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2}m_{\rm Pl}^2 R - \mathscr{L}_{\rm M}\left(\phi_0 + \delta\phi\right)\right]$$

$$ds^2 = a^2(\eta)\left[-(1+2A)d\eta^2 + 2\mathscr{B}_i d\eta dx^i + \left(\delta_{ij} - 2\psi\delta_{ij} + 2\mathscr{E}_{ij}\right)dx^i dx^j\right]$$

There are

1. 5 scalar modes: A, $\mathscr{B}_i \sim B_{,i}$, ψ, $\mathscr{E}_{ij} \sim E_{,ij}$, $\delta\phi$
2. 4 vector modes: $\mathscr{B}_i \sim B_i^{(v)}$, $\mathscr{E}_{ij} \sim E_{i,j}^{(v)}$ (transverse)
3. 2 tensor modes: $\mathscr{E}_{ij} \sim h_{ij}$ (transverse, traceless)

NOT ALL THESE MODES ARE PHYSICAL

How to extract physical degrees of freedom only

Here is the recipe:

1. Eliminate gauge degrees of freedom

2. Construct (gauge invariant) variables and quantize them

3. Calculate what you are interested in: $\mathscr{P}_{\mathscr{R}}$, $n_{\mathscr{R}}$, $f_{\rm NL}$, etc

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   - Fix a gauge: comoving gauge, flat gauge, etc
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   - Popular choices of gauge invariant variables: comoving curvature perturbation $\mathscr{R}$, Sasaki-Mukhanov variable Q
   - Gauge invariant variable $\neq$ good variable
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How to systematically proceed?

Reducing the phase space of $\mathscr{L}$

$$S = \int d^4x \mathscr{L}(q, z)$$

Phase space variables $\{q_\mu, z_I\}$

$$\begin{cases} q_\mu : \text{appears in the kinetic term} \\ z_I : \text{do not appear in the kinetic term} \end{cases}$$

In the Hamiltonian form

$$S = \int d^4x \left[\Pi^\mu q'_\mu - \mathscr{H}\left(\Pi^\mu, q_\mu\right) - \mathscr{C}_I z_I \right]$$

Euler-Lagrange equations of z_I: constraint equations

$$\frac{\partial \mathscr{L}}{\partial z_I} = \mathscr{C}_I = 0 \rightarrow \text{satisfied all the time}$$

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Constraints as gauge transformation generators

In the Einstein gravity, $\{\mathscr{C}_A, \mathscr{C}_B\}_{\text{PB}} = 0$: first class constraints

$$\delta_\xi q_i = \left\{q_i, \xi^\mu \mathscr{C}_\mu\right\}_{\text{PB}} \;;\; \text{generators of gauge transformations}$$

Practically $\mathscr{C}_A$ matters: gauge transformation $\xi^\mu = \left(\xi^0, \xi^{,i}\right)$

$$\mathscr{C}_A \Longleftrightarrow A \sim \delta g_{00} \;:\; \text{energy constraint}$$

$$\mathscr{C}_B \Longleftrightarrow B \sim \delta g_{0i} \;:\; \text{momentum constraint}$$

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Consider scalar metric perturbations: among A, B, ψ and E,

$$\left.\begin{aligned} \widehat{B} - B &= \xi^0 - \xi' \\ \widehat{E} - E &= -\xi \end{aligned}\right\} \to \left(\widehat{B} - \widehat{E}'\right) - \left(B - E'\right) = \xi^0$$

Transformation of canonical variables

Canonical variables are transformed as

$$\delta_\xi\psi = -\mathscr{H}\xi^0, \qquad \delta_\xi\delta\phi = \phi_0'\xi^0, \qquad \delta_\xi E = \xi$$

$$\delta_\xi\Pi^\psi = \frac{2a^2}{\kappa}\nabla^2\xi^0, \quad \delta_\xi\Pi^{\delta\phi} = -a^2\left(3\mathscr{H}\phi_0' + a^2V_\phi\right)\xi^0, \quad \delta_\xi\Pi^E = 0$$

Regarding ξ^0, we can build 2 pairs of canonical variables

$$\mathscr{R} = \psi + \frac{\mathscr{H}}{\phi_0'}\delta\phi \qquad\qquad Q = \delta\phi + \frac{\phi_0'}{\mathscr{H}}\psi$$

$$\Pi^{\mathscr{R}} = \Pi^\psi - \frac{2a^2}{\kappa\phi_0'}\nabla^2\delta\phi \qquad\qquad \Pi^Q = \Pi^{\delta\phi} + \frac{a^3}{\mathscr{H}}\left(\frac{\phi_0'}{a}\right)'\psi$$

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Comoving curvature perturbation — Sasaki-Mukhanov variable

Reduction of a single canonical scalar field (1/2)

Single canonical scalar field in the Einstein gravity

$$
\begin{aligned}
\mathcal{L}_2^{(s)} = \frac{a^2}{2\kappa}\Big[& -6\psi'^2 - 12\mathcal{H}A\psi' - 2\left(\mathcal{H}' + 2\mathcal{H}^2\right)A^2 + 2\left(2A-\psi\right)\nabla^2\psi \\
& + \kappa\left(\delta\phi'^2 + \delta\phi\nabla^2\delta\phi - a^2 V_{\phi\phi}\delta\phi^2\right) + 2\kappa\left(3\phi_0'\psi'\delta\phi - \phi_0' A\delta\phi' - a^2 V_\phi A\delta\phi \right. \\
& \left. + 4\left(\frac{\kappa}{2}\phi_0'\delta\phi - \psi' - \mathcal{H}A\right)\nabla^2\left(B - E'\right)\right]
\end{aligned}
$$

We can build conjugate momenta: Π^ψ, $\Pi^{\delta\phi}$, Π^E

$$
\begin{aligned}
\Pi^\psi &= \frac{a^2}{2\kappa}\left[-12\psi' - 12\mathcal{H}A + 6\kappa\phi_0'\delta\phi - 4\nabla^2\left(B - E'\right)\right] \\
\Pi^{\delta\phi} &= a^2\left(\delta\phi' - \phi_0' A\right) \\
\Pi^E &= -\frac{2a^2}{\kappa}\nabla^2\left(\frac{\kappa}{2}\phi_0'\delta\phi - \psi' - \mathcal{H}A\right)
\end{aligned}
$$

Constructing gauge invariant variables

$$
\begin{aligned}
\mathscr{L}_2^{(s)} = & \Pi^\psi \psi' + \Pi^{\delta\phi}\delta\phi' + \Pi^E E' \\
& - \left\{ \frac{\kappa}{2a^2}\left[\Pi^\psi \Delta^{-2}\Pi^E + \frac{3}{2}\left(\Delta^{-2}\Pi^E\right)^2 + \frac{\Pi^{\delta\phi 2}}{\kappa} \right] + \frac{\kappa}{2}\phi_0'\Pi^\psi\delta\phi \right. \\
& \left. + \frac{a^2}{\kappa}\left[\psi\nabla^2\psi - \frac{3}{4}\kappa^2\phi_0'^2\delta\phi^2 - \frac{\kappa}{2}\left(\delta\phi\nabla^2\delta\phi - a^2 V_{\phi\phi}\delta\phi^2\right)\right]\right\} \\
& - \left[-\mathscr{H}\Pi^\psi + \phi_0'\Pi^{\delta\phi} - \frac{2a^2}{\kappa}\nabla^2\psi + a^2\left(3\mathscr{H}\phi_0' + a^2 V_\phi\right)\delta\phi \right] A - \Pi^E B
\end{aligned}
$$

Constraint equations: $\mathscr{C}_A = \mathscr{C}_B = 0$

$$
\mathscr{R} = \psi + \mathscr{H}\frac{\delta\phi}{\phi_0'} \qquad Q = \delta\phi + \frac{\phi_0'}{\mathscr{H}}\psi
$$

$$
\Pi^{\mathscr{R}} = \Pi^\psi - \frac{2a^2}{\kappa\phi_0'}\nabla^2\delta\phi \qquad \Pi^Q = \Pi^{\delta\phi} + \frac{a^3}{\mathscr{H}}\left(\frac{\phi_0'}{a}\right)'\psi
$$

Reduction of a single canonical scalar field (2/2)

After eliminating $\Pi^{\mathscr{R}}$ / Π^{Q} in favour of $\mathscr{R}'$ / Q',

$$\mathscr{L}_2^{(s)} = \frac{1}{2}\left(\frac{a\phi_0'}{\mathscr{H}}\right)^2\left[\mathscr{R}'^2 - (\nabla\mathscr{R})^2\right]$$

$$= \frac{1}{2}a^2\left\{Q'^2 + \left[\frac{\kappa}{a^2}\left(\frac{a^2\phi_0'^{\,2}}{\mathscr{H}}\right)' - a^2 V_{\phi\phi}\right]Q^2 - (\nabla Q)^2\right\}$$

Reduction of a single canonical scalar field (2/2)

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Precisely equivalent

With $z = a\phi_0'/\mathcal{H}$ and $u = z\mathscr{R} = aQ = a\left(\delta\phi + \phi_0'\psi/\mathcal{H}\right)$

$$\mathcal{L}_2^{(s)} = \frac{1}{2}\left[u'^2 - (\nabla u)^2 + \frac{z''}{z}u\right] \to u_k'' + \left(k^2 - \frac{z''}{z}\right)u_k = 0$$

In addition...

$$\mathcal{L}_2^{(v)} = 0, \quad \mathcal{L}_2^{(t)} = \int d^4x \frac{a^2 m_{\rm Pl}^2}{2}\left(h^{ij\prime}h_{ij}' + h^{ij}\nabla^2 h_{ij}\right)$$

Lagrangian of the Hořava-Lifshitz gravity

In the ADM form, schematically,

$$\mathcal{L}_{\rm HL} = N\sqrt{\gamma}\left[\frac{2}{\kappa^2}\left(K^i{}_jK^j{}_i - \lambda K^2\right) + \mu R + \alpha_1 R^{ij}R_{ij} + \alpha_2 R^2 \right.$$
$$\left. + \alpha_3 \frac{\epsilon^{ijk}}{\sqrt{\gamma}} R_{il}\nabla_j R^l{}_k + \alpha_4 C^{ij}C_{ij} + \sigma \right]$$

$$C^{ij} = \frac{\epsilon^{ikl}}{\sqrt{\gamma}}\nabla_k\left(R^j{}_l - \frac{1}{4}\delta^j{}_l R\right) \;;\; \text{Cotton tensor}$$

$$\mathcal{L}_{\rm M} = N\sqrt{\gamma}\left[\frac{1}{2N^2}\left(\phi' - N^i\phi_{,i}\right)^2 - \sum_{n=1}^{3}\xi_n\partial_i^{(n)}\phi\partial^{i(n)}\phi - V(\phi)\right]$$

Reduced Lagrangian of the HL gravity

$$\mathcal{L}_2^{(s)} = \Pi^Q Q' - \mathscr{A}_Q + \frac{\mathscr{B}_Q^2}{4\mathscr{F}_Q} = \Pi^{\mathscr{R}} \mathscr{R}' - \mathscr{A}_{\mathscr{R}} + \frac{\mathscr{B}_{\mathscr{R}}^2}{4\mathscr{F}_{\mathscr{R}}}$$

$$= \frac{1}{2} u'^2 - u \left[\left(\frac{\mathscr{G}_1'}{4\mathscr{G}_1} \right)' - \frac{\mathscr{G}_1'^2}{8\mathscr{G}_1^2} - \mathscr{G}_1 \left(\frac{\mathscr{G}_2}{2\mathscr{G}_1} \right)' - \frac{1}{2}\mathscr{G}_2^2 + 2\mathscr{G}_1\mathscr{G}_3 \right] u$$

$$\mathscr{F}_* \sim \nabla^6 \text{ ; dominant in the UV regime}$$

$$\mathcal{L}_2^{(v)} = 0$$

$$\mathcal{L}_2^{(t)} = \Pi^{ij} h_{ij}' - \mathscr{H}^{(t)}$$

- Einstein gravity results in the IR limit
- Additional constraint $\mathscr{C}_2$ from $\{\mathscr{H}^{(s)}, \mathscr{C}_A\}_{PB}$: Meaning?
- No dynamical vector perturbations

Cubic order Lagrangian in the Einstein gravity

$$
\begin{aligned}
\mathcal{L}_2 = a^2 \Big\{ & 6(A-\psi)\psi'^2 + 6\mathcal{H}\left(3A^2+2A\psi-\psi^2\right)\psi' + 3\mathcal{H}^2 A^2 (A+\psi)^2 (5A-\psi) + 4\left[\psi' + \mathcal{H}(A-\psi)\right]\psi^{,i}\mathcal{B}_i \\
& + \left[2\psi'^2 - 4\mathcal{H}(A-\psi)\psi' - 3\mathcal{H}^2\left(3A^2+2A\psi-\psi^2\right)\right]\mathcal{E}^i{}_i + 2\left[2(A-\psi)\psi' + \mathcal{H}\left(3A^2+2A\psi-\psi^2\right)\right] \\
& \times \left(\mathcal{B}^i{}_{,i} - \mathcal{E}^{i\prime}{}_i\right) - 3\mathcal{H}\left[2\psi' + \mathcal{H}(3A+\psi)\right]\mathcal{B}^i\mathcal{B}_i - 8\mathcal{H}\psi^{,i}\mathcal{B}^j\mathcal{E}_{ij} + 4\mathcal{H}\psi^{,i}\mathcal{B}_i\mathcal{E}^j{}_j \\
& - \mathcal{H}\left[2\psi' - 3\mathcal{H}(A-\psi)\right]\left(\mathcal{E}^i{}_i\right)^2 + 2\mathcal{H}\left[2\psi' - 3\mathcal{H}(A-\psi)\right]\left(\mathcal{E}_{ij}\right)^2 + 4\left[\psi' + 2\mathcal{H}(A-\psi)\right]\mathcal{E}^{ij} \\
& \times \left(\mathcal{B}_{(i,j)} - \mathcal{E}'_{ij}\right) + 4\left[\psi' + \mathcal{H}(A-\psi)\right]\mathcal{B}^i\left(2\mathcal{E}_{ij}{}^{,j} - \mathcal{E}^j{}_{j,i}\right) + 4\psi^{,i}\mathcal{B}^j\left(\mathcal{B}_{(i,j)} - \mathcal{E}'_{ij}\right) \\
& + (A-\psi)\left[\left(\mathcal{B}^i{}_{,i} - \mathcal{E}^{i\prime}{}_i\right)^2 - \left(\mathcal{B}_{(i,j)} - \mathcal{E}'_{ij}\right)^2\right] + 3\mathcal{H}^2\left(\mathcal{B}^i\mathcal{B}_i\mathcal{E}^i{}_i - 2\mathcal{B}^i\mathcal{B}^j\mathcal{E}_{ij}\right) - 2\mathcal{H}\mathcal{B}^i\mathcal{B}_i\left(\mathcal{B}^i{}_{,i} - \mathcal{E}^{i\prime}{}_i\right) \\
& + \mathcal{H}^2\left[6\left(\mathcal{E}_{ij}\right)^2\mathcal{E}^k{}_k - \left(\mathcal{E}^i{}_i\right)^3 - 8\mathcal{E}^{ij}\mathcal{E}_{jk}\mathcal{E}^k{}_i\right] \\
& + 4\mathcal{H}\mathcal{B}^i\left[2\mathcal{E}_{ij}\left(2\mathcal{E}^{jk}{}_{,k} - \mathcal{E}^k{}_k{}^{,j}\right) + 2\mathcal{E}^{jk}\left(\mathcal{E}_{ij,k} - \mathcal{E}_{jk,i} + \mathcal{E}_{ki,j}\right) - \left(2\mathcal{E}_{ij}{}^{,j} - \mathcal{E}^j{}_{j,i}\right)\mathcal{E}^k{}_k\right] \\
& + 2\mathcal{H}\left[\left(\mathcal{E}^i{}_i\right)^2 - 2\left(\mathcal{E}_{ij}\right)^2\right]\left(\mathcal{B}^i{}_{,i} - \mathcal{E}^{i\prime}{}_i\right) + 8\mathcal{H}\mathcal{E}^{ij}\left[-\mathcal{E}^k{}_k\left(\mathcal{B}_{(i,j)} - \mathcal{E}'_{ij}\right) + 2\mathcal{E}_j{}^k\left(\mathcal{B}_{(k,i)} - \mathcal{E}'_{ki}\right)\right] \\
& + 2\mathcal{B}^i\left[\left(2\mathcal{E}_{ij}{}^{,j} - \mathcal{E}^j{}_{j,i}\right)\left(\mathcal{B}^k{}_{,k} - \mathcal{E}^{k\prime}{}_k\right) - \left(\mathcal{E}_{ij,k} - \mathcal{E}_{jk,i} + \mathcal{E}_{ki,j}\right)\left(\mathcal{B}_{(j,k)} - \mathcal{E}'_{jk}\right)\right] \\
& + 4\mathcal{E}^{ij}\left[\left(\mathcal{B}_{(i,j)} - \mathcal{E}'_{ij}\right)\left(\mathcal{B}^k{}_{,k} - \mathcal{E}^{k\prime}{}_k\right) - \left(\mathcal{B}^{(k}{}_{,j)} - \mathcal{E}^{k\prime}{}_j\right)\left(\mathcal{B}_{(i,k)} - \mathcal{E}'_{ik}\right)\right] + \textbf{32 more terms}\Big\}
\end{aligned}
$$

Alternative description

The remaining procedures are conceptually easy

1. Calculate the conjugate momenta of canonical variables
2. Reduce the Lagrangian in the Hamiltonian form
3. Solve the constraint equations
4. ☺

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However practically

We from the beginning start with canonical form

$$\mathscr{L}_{\rm gr} = \Pi^{ij}\gamma'_{ij} - N\sqrt{\gamma}\left(-R + \gamma^{-1}\Pi^{ij}\Pi_{ij} - \frac{1}{2}\gamma^{-1}\Pi^2\right) + 2N_i\sqrt{\gamma}\nabla_j\left(\gamma^{-1/2}\Pi^{ij}\right)$$

Conclusions

1. Gauge issue is persistent in cosmological perturbations
   Freedom of coordinate choice
2. Hamiltonian reduction of cosmological perturbations
   - Systematically reduce the phase space
   - We are left with gauge invariant, dynamical variables
3. Wide applications to many (exotic) models
   Higher order gravity, $f(R)$ gravity, vector field background, etc